Abelianization of G-Higgs bundles

07/06/16

0. Motivation

G reductive $/ k = \bar{k}$ $\quad$ $\mathrm{char}(k) \nmid |W|$ $\quad$ where $W = W(G)$

X smooth proj curve $/ k$

Last week: $\mathcal{H}iggs_G \longleftrightarrow$ sheaf of Picard cat^s / Hitchin base

for $G = Gl_n, Sl_n, PSl_n$.

Basic idea for $G = Gl_n$: Higgs bdle $\rightsquigarrow$ vbdle + endomorphism

$\rightsquigarrow$ diagonalise, fixing char. polynomial

$(E, \varphi) \rightsquigarrow \begin{matrix} \tilde{X}_a \\ \downarrow \pi \\ X \end{matrix}$ spectral curve

$\rightsquigarrow$

eigenbdle $\mathcal{L} = \ker(\pi^*\varphi - \lambda \mathrm{id}) \subset \pi^* E$

line bdle for $\tilde{X}_a$ integral

Char. polynomial ($\hat{=}$ spectral curve) $\longleftrightarrow$ unordered set of eigenvalues

$\rightsquigarrow$

ordered ———//———

$\hat{=}$ diagonal matrices

$\hat{=}$ Cartan subalgebra $\mathfrak{t} \otimes \underset{\text{canonical bdle}}{\mathcal{K}}$

$$\begin{tikzcd} \tilde{X}_a \arrow[r] \arrow[d] & \mathfrak{t} \otimes \mathcal{K} \arrow[d, "\text{"cameral cover"}"] \\ X \arrow[r] & \mathfrak{t}/W \otimes \mathcal{K} \end{tikzcd}$$

Eigenbdle $\longleftrightarrow$ principal T-bundle $\mathcal{L}$

$\mathrm{rk}\, \mathcal{L} = 1 \longleftrightarrow$ eigenspaces of $\pi^*\varphi$ 1-dim.

Def A matrix M is called <u>regular</u> if its eigenspaces are 1-dim

($\Leftrightarrow$ its Ad-orbit is max-dim. $\Rightarrow C_{Gl_n}(M)$ is abelian)

1. Setup

X/k smooth proj. $\qquad \deg L \geq 2g-1$ or $L = K_X$

$f : X \longrightarrow \operatorname{Spec} k$ $\qquad \tilde{G} := G \times X \longrightarrow X$

$L \to X$ line bdle

$$f_*[\mathfrak{g}\otimes L/\tilde{G}] \xrightarrow[h_{G,L}]{} H^0(X, \mathfrak{t}/W \otimes L) \qquad \text{Hitchin map}$$

$$\| $$

$$\mathcal{H}iggs_{G,L}$$

induced by $\chi : \mathfrak{g} \longrightarrow \mathfrak{t}/W$

$$[\chi]_L : [\mathfrak{g}\otimes L/G] \longrightarrow \underbrace{\mathfrak{t}/W}_{=:\mathfrak{c}} \otimes L$$

$$\begin{tikzcd}
{[\chi]_{\mathbb{G}_m} : [\mathfrak{g}/G\times\mathbb{G}_m]} \arrow[r] \arrow[dr] & {[\mathfrak{c}/\mathbb{G}_m]} \arrow[d] \\
X \arrow[r, "{[L]}"'] & B\mathbb{G}_m
\end{tikzcd}$$

pullback by $[L]$ to get $[\chi]_L$!

2. The stack $[\mathfrak{g}/G] \xrightarrow{[\chi]} \mathfrak{c}$

Prop. Let $\mathfrak{g}_{reg} := \{x \in \mathfrak{g} \mid Ad(G)(x) \text{ max-dim.}\}$.

Then $[\chi^{reg}] := [\chi]|_{[\mathfrak{g}^{reg}/G]} : [\mathfrak{g}^{reg}/G] \longrightarrow \mathfrak{c}$

is a gerbe. Moreover, $\exists\ J \longrightarrow \mathfrak{c}$ affine gp scheme, smooth, s.th. $[\mathfrak{g}_{reg}/G] = BJ$.

Proof.

Gerbe means
- locally nonempty
- locally connected.

Facts (Kostant):

(i) χ_{reg} is smooth.

(ii) fibres of χ_{reg} are G-orbits.

(iii) $\exists$ section $s_K : \mathfrak{c} \to \mathfrak{g}_{reg}$ ("Kostant section").

For sl_n:

$$\begin{array}{ccc} \mathfrak{c} \cong k^n & \longrightarrow & sl_n \\ \Psi & & \Psi \\ (a_2, \dots, a_n) & \longmapsto & \begin{pmatrix} 0 & a_2 & a_3 & \cdots & a_n \\ 1 & \ddots & \ddots & \ddots & \vdots \\ & \ddots & \ddots & \ddots & a_3 \\ & & \ddots & \ddots & a_2 \\ 0 & & & 1 & 0 \end{pmatrix} \end{array}$$

In general: Choose $sl_2 \longrightarrow \mathfrak{g}$

$\frac{1}{2}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \longmapsto x = 2 \sum_{\alpha \in \Phi^+} \alpha^\vee$ — positive roots

$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \longmapsto e \in \mathfrak{g}_{reg, nilp}$

$e = \sum_{\alpha \in \Delta} x_\alpha$, Δ: simple positive roots; x_α: generator of $\mathfrak{g}_\alpha$

Complete to sl_2-triple x, e, f.

$\chi_{reg} : f + C_{\mathfrak{g}}(e) \xrightarrow{\sim} \mathfrak{c}$ iso

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Local (even global nonemptiness): $\mathfrak{c} \longrightarrow [\mathfrak{g}_{reg}/G]$

given by $\left(\mathfrak{c} \times G,\ \begin{array}{l} \mathfrak{c} \times G \longrightarrow \mathfrak{g}_{reg} \\ (x,g) \mapsto \mathrm{Ad}(g^{-1})(s_K(x)) \end{array}\right)$

is a global section.

Hence if this is a J-gerbe, then it is neutral, ie $\simeq BJ$.

Local connectedness:

$$S \xrightarrow{a} \mathfrak{c}$$

$$(E,\varphi),\ (E',\varphi') \in [\mathfrak{g}^{reg}/G](S)$$

wlog $E \to G \times S$ trivial
$\ E' \to$

Consider

$$\begin{tikzcd} (\varphi,\varphi') : S \arrow[r] \arrow[dr, dashed, "\text{lifts locally! (by smoothness)}"'] & \mathfrak{g}^{reg} \times_{\mathfrak{c}} \mathfrak{g}^{reg} & \ni (x, Ad(g)(x)) \\ & G \times \mathfrak{g}^{reg} \arrow[u, "\text{smooth}"'] & \ni (g,x) \arrow[u, mapsto] \end{tikzcd}$$

J-gerbe:

$$\begin{tikzcd} I := \{(x,g) \mid Ad(g)(x) = x\} \subset \mathfrak{g}^{reg} \times G \arrow[rr] \arrow[dr] \arrow[dd] & & \exists J \text{ descent of } I \arrow[dl] \arrow[dd] \\ & \exists J \text{ descent of } I \arrow[dd] & \\ \mathfrak{g}^{reg} \arrow[rr] \arrow[dr, bend right] & & [\mathfrak{g}^{reg}/G] \arrow[dl, bend left] \\ & \mathfrak{c} & \end{tikzcd}$$

$\Longrightarrow$ J-gerbe $\square$

3. Back to $[\mathfrak{g}^{reg}/G \times \mathbb{G}_m]$

<u>Prop.</u> $[\chi_{reg}]_{\mathbb{G}_m} = [\mathfrak{g}^{reg}/G \times \mathbb{G}_m] \longrightarrow [\mathfrak{c}/\mathbb{G}_m]$ is a J-gerbe

(NOT necessarily neutral!)

$\square$

Proof. $\mathbb{G}_m \circlearrowright I$, so use descent as before.

Why isn't it neutral?

Kostant section $f + c_{\mathfrak{g}}(e)$ is not $\mathbb{G}_m$-invariant.

Define $\rho = 2\sum \alpha^\vee : \mathbb{G}_m \longrightarrow T$

Infinitesimal action: $\underset{\{x,e\}}{sl(2)} \subset \mathfrak{g} = \bigoplus_i V_i$

$\leftarrow$ irreducible subrep^s

highest wt vectors e_i generate $c_{\mathfrak{g}}(e)$

$ad(x)(e_i) = m_i e_i$

$\Rightarrow \deg(u_i) = m_i + 1$

where the u_i are homogeneous generators of $k[\mathfrak{t}]^W$

$$\mathbb{G}_a \xrightarrow{\delta} End(\mathfrak{g})$$
$$\lambda \longmapsto ad(\lambda x)$$

$\rightsquigarrow \delta(\lambda)(f + \sum_i \lambda_i e_i) = -\lambda f + \sum \lambda_i m_i \lambda e_i$

$\delta' := \delta + \lambda id \rightsquigarrow \delta'$ preserves $f + c_{\mathfrak{g}}(e)$

We get $\mathbb{G}_m \longrightarrow Aut\,\mathfrak{g} \times \mathbb{G}_m$

Problem: Doesn't lift to $\mathbb{G}_m \longrightarrow G \times \mathbb{G}_m$!

But after taking a square root it does:

$$\begin{tikzcd} \mathbb{G}_m \arrow[r, dashed, "\exists \tilde{\rho}"'] \arrow[d, "(-)^2"'] & G \times \mathbb{G}_m \arrow[d] \\ \mathbb{G}_m \arrow[r] & Aut\,\mathfrak{g} \times \mathbb{G}_m \end{tikzcd}$$

Hence put
$$\begin{tikzcd} {[G/\mathbb{G}_m]^{(2)}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {[G/\mathbb{G}_m]} \arrow[d] \\ B\mathbb{G}_m \arrow[r, "(-)^2"] & B\mathbb{G}_m \end{tikzcd}$$

similarly $\left[\mathfrak{g}^{reg}/G\times\mathbb{G}_m\right]^{(2)}$.

We get a section

$$\begin{tikzcd}[column sep=large]
{[\mathfrak{g}_{reg}/G \times \mathbb{G}_m]^{(2)}} \arrow[r, shift left] & {[C/\mathbb{G}_m]^{(2)}} \arrow[l, dashed, shift left, "\exists"]
\end{tikzcd}$$

$$\rotatebox[origin=c]{90}{$\in$} \qquad\qquad\qquad\qquad \rotatebox[origin=c]{90}{$\in$}$$

$$\left(\begin{tikzcd}[column sep=small]
\tilde{p}(L^{1/2}) \arrow[rr] \arrow[dr, "s_{n \circ f}"'] & & \mathfrak{g}_{reg} \\
& f + s_g(e) \arrow[ur] &
\end{tikzcd} \right) \longleftarrow\!\!\shortmid \left(\begin{matrix} L^{1/2} \\ \downarrow \\ S \end{matrix},\ L \xrightarrow[f]{\mathbb{G}_m} C \right)$$

<u>Detour</u> $[\mathfrak{g}_{reg,ss}/G]$:

$$\mathfrak{t}_{reg} = \{ x \mid \alpha(x) \neq 0 \ \forall \alpha \text{ root} \}$$

$$Ad(G)(\mathfrak{t}_{reg}) = \mathfrak{g}_{reg,ss} \qquad\qquad N = N_G(T) \text{ normalizer}$$

<u>Prop</u> $[\mathfrak{t}_{reg}/N] \xrightarrow{\sim} [\mathfrak{g}_{reg,ss}/G]$ equivalent gerbes over $C_{reg} = \mathfrak{t}_{reg}/W$

Proof. See $[\mathfrak{t}_{reg}/N]$ as a J-gerbe.

Local nonemptiness: Étale-locally $\mathfrak{t}_{reg} \simeq \mathfrak{t}_{reg}/W \times W$

Local connectedness: Follows from that of $[\mathfrak{g}_{reg}/G]$

and $(x,y) \in \mathfrak{t}_{reg} \times_{C_{reg}} \mathfrak{t}_{reg}$

$\Rightarrow x, y$ N-conjugate

and $\mathfrak{z}_{\mathfrak{g}}(x) = \mathfrak{t}$ by regularity

J-gerbe:

$$\begin{tikzcd}
& \mathfrak{t}_{reg} \times T \arrow[r] \arrow[d] \arrow[dl] & I \arrow[d] \arrow[dr] & \\
& \mathfrak{t}_{reg} \arrow[r, "j"] \arrow[d] & \mathfrak{g}_{reg} \arrow[d] & J \arrow[dl] \\
J' \hookrightarrow i^*J \arrow[r] & C_{reg} \arrow[r, "i"] & C &
\end{tikzcd}$$

$$\mathcal{J}|_{S_{reg}}(U \to S_{reg}) = \mathrm{Hom}_W(U \times_{S_{reg}} A_{reg}, T)$$

$\square$